The Corona Theorem on the Complement of Certain Square Cantor Sets

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Abstract

Let K be a square Cantor set, i.e. the Cartesian product $K = E \times E$ of two linear Cantor sets. Let δ_n denote the proportion of the intervals removed in the nth stage of the construction of E. It is shown that if $\delta_n = o(\frac{1}{\log \log n})$ then the corona theorem holds on the domain $\Omega = \mathbb{C}^* \setminus K$.

1 Introduction

Denote by $H^\infty(\Omega)$ the collection of bounded analytic functions on a plane domain Ω. Let us denote by $\mathcal{M} = \mathcal{M}(H^\infty(\Omega))$ the set of multiplicative linear functionals on $H^\infty(\Omega)$, i.e. $\mathcal{M}$ is the maximal ideal space of $H^\infty(\Omega)$. When $H^\infty(\Omega)$ separates the points of Ω, there is a natural identification of Ω with a subset of Ω through the functionals defined by pointwise evaluations. The corona problem for $H^\infty(\Omega)$ is to determine whether $\mathcal{M}$ is the closure of Ω in the Gelfand topology.

The problem can be stated more concretely as follows. Given $\{f_j\}_1^n \in H^\infty(\Omega)$ and $\delta > 0$ such that $1 \geq \max_j |f_j(z)| > \delta$ for all $z \in \Omega$, do there exist $\{g_j\}_1^n \in H^\infty(\Omega)$ such that

$$\sum_{j=1}^{n} f_j g_j = 1?$$

We call the functions $\{f_j\}$ corona data, and the functions $\{g_j\}$ corona solutions. It has been conjectured that this can be answered in the affirmative for any planar domain. In the case of Riemann surfaces, a counterexample was first found by Brian Cole (see [16]), other examples being found later by D. E. Barrett and J. Diller [3], however, this is due to a structure that in some sense makes the surface seem higher dimensional, so one might hope that the restriction to the Riemann sphere might prevent this obstacle.

This problem was first posed in the case of the unit disc $\mathbb{D}$ by S. Kakutani in 1941, which case was solved by L. Carleson [8] in 1962. It is from this origin that the problem gets its name, as there would have been a set of maximal ideals suggestive of the sun's corona has the theorem failed in this case. The proof was subsequently simplified by L. Hörmander [22] by use of a $\bar{\partial}$-problem, and then later in an acclaimed proof by T. Wolff (see [17] or [19]). A quite different proof of this result using techniques from several complex variables was later developed by B. Berndtsson and T. J. Ransford [6] and Z. Slodkowski [27]. Besides the intrinsic interest of this result in classical function theory, the proof of this result introduced a number of tools and ideas which have proven to be of great importance in analysis.

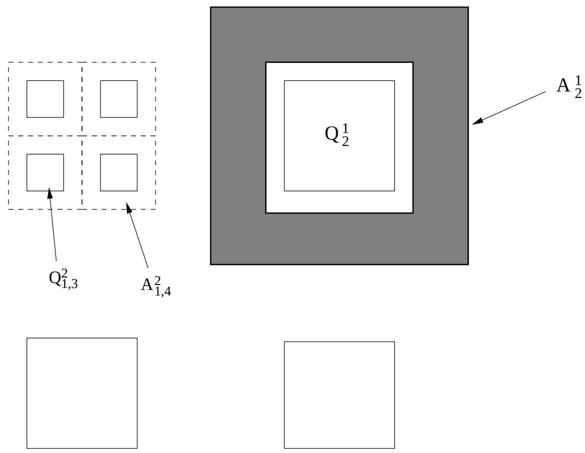

Figure 1: The various geometric constructs associated to the Cantor set: Q_J^n, V_J^n, and A_J^n.

Following the appearance of this result, it was swiftly generalized to the case of finitely connected domains, and by now a number of proofs exist for this case, e.g. [1], [2], [12], [13], [28], [26], [29], [30]. Although any planar domain can be exhausted by a sequence of finitely connected domains, to solve the problem on the larger domain one must control the norms of the corona solutions, $\|g_j\|_\infty$, in the approximating domains in order to take normal limits, control which is unfortunately not provided by any of the proofs just cited.

The corona problem was first solved for a class of infinitely connected domains by M. Behrens [4], [5]. These are "roadrunner" domains $\mathbb{D} \setminus \bigcup B_j$, where $B_j = B(c_j, r_j)$ is a disc centered at c_j and radius r_j such that

$$\sum_{j=1}^{\infty} \frac{r_j}{|c_j|} < \infty \qquad \text{and} \qquad \left|\frac{c_{j+1}}{c_j}\right| < \lambda < 1 \text{ for all j.}$$

This summability restriction was improved somewhat in [10], [11]. Behrens [5] also proved that if the corona theorem fails for a plane domain then it fails for a domain of the form $\mathbb{D} \setminus \bigcup B_j$, where $\{B_j\}$ is a sequence of discs clustering only at the origin. In this direction there is also a result of Gamelin [14] that the corona problem is local in that it depends only on the behavior of the domain locally about each boundary point.

The next significant progress for infinitely connected domains was again achieved by Carleson [9], who solved the corona problem for domains having boundary $E \subset \mathbb{R}$ satisfying, for some $\epsilon > 0$,

$$\Lambda_1(B(x,r) \cap E) \geq \epsilon r$$

for every $x \in E$ and $r > 0$, where Λ_1 denotes one-dimensional Hausdorff measure. This proof followed an idea introduced in [13], constructing an explicit projection from $H^\infty(\mathbb{D})$ onto $H^\infty(\Omega)$. Following in the same vein, P. Jones and D. Marshall [23] used such projections to show that if the corona problem can be solved at the critical points of Green's function for the domain, then it can be solved for the domain, and they provided a number of sufficient conditions for this criterion.

Later J. Garnett and P. Jones [20] extended the result of Carleson [9], showing that the corona theorem holds for any domain having boundary contained in $\mathbb{R}$. This was later extended by C. Moore

[25] to the case of domains with boundary contained in the graph of a $C^{1+\epsilon}$ function.

Due to the results in [23], the corona problem for a domain can be solved if the critical points of Green's function (for a fixed base point) form an interpolating sequence for $H^\infty(\Omega)$. However, it was shown by M. Gonzalez [21] that for a large class of domains, conditions necessary for the critical points to be an interpolating sequence might fail. Peter Jones (unpublished) has given an explicit example of a class of domains where the critical point of greens function fail to form an interpolating sequence, namely the complements of certain square Cantor sets (which we describe more explicitly below).

Throughout the remainder of this work, Ω will denote the complement of a square Cantor set K. To fix notation, we define K explicitly as $K = \bigcap_n K^n$, where the K^n are defined inductively as follows. Fix $\{\lambda_n\}_{\mathbb{N}} \in (\frac{1}{4}, \frac{1}{2})$, which we assume satisfies $\lambda_n \leq \lambda_{n+1}$ for simplicity. (These conditions also ensure that $H^\infty(\Omega)$ is nontrivial.) Put $K^0 = [0,1]^2$. At stage n, we set $K^n = \bigcup_{|J|=n} Q^n_J$ where Q^n_J are squares with sides parallel to the axes and side length $\sigma_n = \prod_{k=1}^n \lambda_k$, and J is a multi-index of length $|J| = n$ on letters $\{1,2,3,4\}$. At stage $n+1$, we construct squares $Q^{n+1}_{J,j} \subset Q^n_J$, $j \in \{1,2,3,4\}$, of side length σ_{n+1} with sides parallel to the axes such that each $Q^{n+1}_{J,j}$, $1 \leq j \leq 4$, contains a corner of Q^n_J. We define $K^{n+1} = \bigcup_{|J|=n} \bigcup_{j=1}^4 Q^{n+1}_{J,j}$.

Some auxiliary definitions are also useful in this setting. The first of these is the quantity $\delta_n = 1-2\lambda_n$, which represents the normalized gap width between squares of the nth generation having common parent. It is useful also to introduce the thickened squares $V^n_J := (1+\frac{\delta_n}{2\lambda_n})Q^n_J$, and from these the "square annuli" A^n_J given by $A^n_J = \overline{V^n_J \setminus \bigcup_{j=1}^4 V^{n+1}_{J,j}}$. We note that our constants were chosen so that $\bigcup_1^4 V^{n+1}_{J,j}$ is a single square concentric with and containing Q^n_J. That $A^n_J \neq \emptyset$ follows from the assumption that $\lambda_n \leq \lambda_{n+1}$. Let us denote by z^n_J the center of the square Q^n_J. (See Figure 1.)

For the harmonic measures below, we will generally make the abbreviations $\omega(\cdot) := \omega(\infty, \cdot, \Omega)$ and $\omega(\cdot, z) := \omega(z, \cdot, \Omega)$ or $\omega_z(\cdot) := \omega(z, \cdot, \Omega)$ when there is no risk of confusion.

We briefly sketch the proof of the result of Jones. Let us denote by $\{z_j\}$ the critical points of Green's function $g(z) = g(z, \infty)$ for the domain Ω. For $\{z_j\}$ to be an interpolating sequence, it is necessary that the sum $\sum g(z_j)$ be finite. Roughly speaking, each square annulus A^n_J contains one critical point, and $g(z)$ is of the same size as $\omega(Q^n_J)$ (up to a constant factor) for $z \in A^n_J$ when there are $0 < a < b < \frac{1}{2}$ such that $a \leq \lambda_n \leq b$ for every n. Thus by Harnack's inequality the convergence of $\sum g(z_j)$ is equivalent to the convergence of $\sum_{n,J} \omega(Q^n_J)$. But $\sum_{n,J} \omega(Q^n_J) = \sum_n \omega(K) = \infty$, so $\{z_j\}$ cannot be an interpolating sequence.

If one takes slightly more care in comparing harmonic measure to Green's function, this can be extended to show that $\{z^n_J\}_{n,J}$ is not an interpolating sequence when $\delta_n \leq \frac{c_0}{\log n}$ for large n, where c_0 is some absolute constant.

In the other direction, it has been shown by Jones (unpublished) that the critical points of Green's function form an interpolating sequence in the case that K has positive area. (This occurs iff $\sum \delta_n < \infty$.) This can be done via the techniques of [23] using harmonic measure estimates (see Theorems 4.1 and 4.5 of that paper), or, alternatively, by solving a $\bar{\partial}$-problem.

In the present work, we prove the following result.

Theorem 1.1. *If $K = K(\{\lambda_n\})$ is a square Cantor set with $\delta_n = o\left(\frac{1}{\log\log n}\right)$ and $\{f_\mu\}_{\mu=1}^M \in H^\infty(\Omega)$ satisfy*

$$0 < \eta \leq \max_{1 \leq \mu \leq M} |f_\mu(z)| \leq 1,$$

then there are $\{g_\mu\}_{\mu=1}^M \in H^\infty(\Omega)$ *such that*

$$\sum_{\mu=1}^{M} f_\mu(z) g_\mu(z) = 1, \qquad z \in \Omega.$$

This extends the result past the regime where the critical points of Green's function form an interpolating sequence.

The basic outline of the proof is as follows. We first break the domain into a number of simply connected domains T_J^n which, roughly speaking, are the cross-shaped domains $Q_J^n \setminus \bigcup_{j=1}^4 Q_{J,j}^{n+1}$. The fundamental idea is that found in [20], namely to apply Carleson's original theorem for simply connected domains and then constructively solve a $\bar{\partial}$-problem to obtain corona solutions for the whole domain.

To obtain the necessary cancellations, however, we require that the corona solutions on our simply connected subdomains have a certain amount of agreement on the overlaps of those domains. To achieve this, we build these special solutions inductively, the solutions at stage n obtained first by choosing solutions according to Carleson's corona theorem for simply connected domains and then solving a $\bar{\partial}$-problem to alter the solutions to match the neighboring solutions already constructed. The method for solving the $\bar{\partial}$-problem is much like in [20], employing an interpolating sequence in the simply connected domains T_J^n to build our solutions. For our solution to have the desired special properties, however, we must choose our interpolating functions to have certain special properties, and it is here that we need the condition that $\delta_n = o(\frac{1}{\log \log n})$. Section 3 is devoted to constructing these interpolating functions and solving the associated $\bar{\partial}$-problem in our simply connected subdomains.

Once these solutions have been constructed we paste these solutions together by solving another $\bar{\partial}$-problem. In this case, the $\bar{\partial}$-problem is solved using the ideas of rational approximation theory (see [32] or Chapter XII of[15]). Essentially, one solves the $\bar{\partial}$-problem on the intersection of two subdomains by the usual Cauchy integral representation, but in order to be able to sum these various pieces, we must add additional cancellation to each piece. This is done by subtracting a bounded analytic function on Ω which has singular support on a portion of the Cantor set nearby and which matches derivatives of the integral at infinity. Schwarz lemma bounds, in conjunction with the cancellations from our special corona solutions in the subdomains, then allow us to sum the terms and obtain the solution to the $\bar{\partial}$-problem. This part of the proof implicitly makes use of the fact that for any $\zeta \in K$ there is $c_0 > 0$ such that

$$\gamma(B(\zeta, r) \cap K) \geq c_0 r$$

for $r \in (0, \operatorname{diam}(K)]$, where

$$\gamma(E) := \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}^* \setminus E), \|f\| \leq 1\}$$

denotes the analytic capacity of the set E (see [24]). This parallels the thickness condition for the boundary employed in [9]. Thus the functions introduced to obtain cancellations are polynomials in the extremal function for this problem, the Ahlfor's function, for a piece of the boundary. Alternatively, one can take the function to be powers of the Cauchy integral of the uniform measure on an appropriate subsquare $K_J^n := K \cap Q_J^n$ of the Cantor set (see [18]).

The present work is part of the author's Ph.D. dissertation. The author is naturally greatly indebted to his advisor, John Garnett, for many helpful conversations and suggestions over the years, and would like to take this opportunity to express his gratitude. The author would also like to thank Peter Jones for helpful conversations.

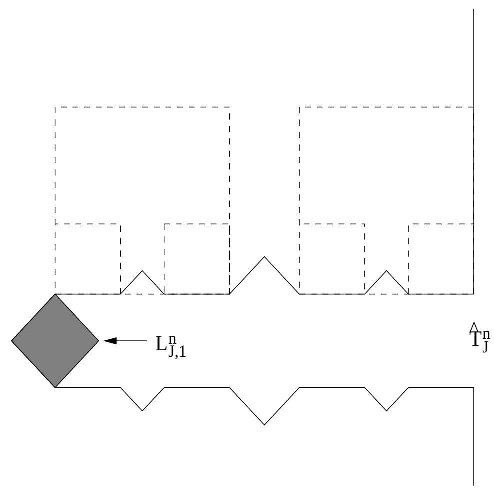

Figure 2: One of the domains $\hat{T}^n_J$.

2 Proof of the Theorem

To begin the proof, let us first define simply connected, cross-shaped regions T^n_J by $T^n_J := Q^n_J \setminus \bigcup_1^4 Q^{n+1}_{J,j}$. For these regions we distinguish the four boundary segments, denoted $\{\ell^n_{J,j}\}_{j=1}^4$, which do not lie on a side of a square $Q^{n+1}_{J,j}$, $1 \leq j \leq 4$. Let us then define lozenges

$$L^n_{J,j} = \left\{ z : \left| \arg \frac{z - x_2}{x_2 - x_1} \right| \vee \left| \arg \frac{z - x_1}{x_2 - x_1} \right| < \alpha \right\}, \tag{1}$$

where $(x_1, x_2) = \ell^n_{J,j}$ and α is some angle less than $\frac{\pi}{4}$ to be determined by the constructions below. We now define regions

$$\hat{T}^n_J = T^n_J \cup \bigcup \{L^m_{I,i} : \ell^m_{I,i} \subset \partial T^n_J\}.$$

(See Figure 2.) We note that these regions remain simply connected, so that Carleson's corona theorem for the unit disc $\mathbb{D}$ provides corona solutions $\{g^{(n,J)}_\mu\}_{\mu=1}^M$ corresponding to $\{f_\mu\}_1^M$ in $\hat{T}^n_J$. We note that $\sum \chi_{\hat{T}^n_J} \leq 2$. The most important result for the present construction is embodied in the following proposition.

Proposition 2.1. *Given corona data $\{f_\mu\}_1^M$ in Ω as above, if the angle α defining the lozenges $L^n_{J,j}$ is sufficiently small then there are corona solutions $\{g^{(n,J)}_\mu\}_1^M$ in each $\hat{T}^n_J$ such that $\|g^{(n,J)}_\mu\|_\infty \leq C(M, \delta, K)$ and if (n, J) and (m, I) are indices with $\hat{T}^n_J \cap \hat{T}^m_I = L^n_{J,j}$ then for $1 \leq \mu \leq M$,*

$$|g^{(n,J)}_\mu(z) - g^{(m,I)}_\mu(z)| \lesssim \frac{1}{n^3} \frac{d(z,K)}{\sigma_n \delta_n}, \qquad z \in L^n_{J,j}.$$

Assuming this proposition, let us prove the theorem. Let $\{\phi_{(n,J)}\}$ to be a partition of unity on Ω subordinate to $\hat{T}^n_J$ satisfying $|\nabla\phi_{(n,J)}(z)| \lesssim d(z,K)^{-1}$ and define

$$\tilde{g}_\mu = \sum_{(n,J)} \phi_{(n,J)} g_\mu^{(n,J)}. \tag{2}$$

We note that the functions $\tilde{g}_\mu$, while not analytic, have the desired property that $\sum f_\mu \tilde{g}_\mu \equiv 1$ on Ω. An observation due to Hörmander [?], now reduces us to solving the $\bar{\partial}$-problem

$$\bar{\partial} a_{\mu\nu} = \tilde{g}_\mu \bar{\partial}\tilde{g}_\nu, \qquad a_{\mu\nu} \in L^\infty(\Omega).$$

Indeed, given such functions $a_{\mu\nu}$, if we define

$$G_\mu = \tilde{g}_\mu + \sum_{\nu=1}^{M} (a_{\mu\nu} - a_{\nu\mu}) f_\nu$$

the antisymmetry of the matrix $A = [a_{\mu\nu}]$ gives us $\sum f_\mu G_\mu \equiv 1$ on Ω, while (2) provides $\bar{\partial} G_\mu \equiv 0$ on Ω for each $1 \leq \mu \leq M$, so that $G_\mu \in H^\infty(\Omega)$. Thus the functions $\{G_\mu\}_1^M$ provide the corona solutions sought by the theorem.

Let us therefore turn our attention to solving (2). The immediate thought is to consider

$$a_{\mu\nu}(z) = \frac{1}{2\pi i} \iint_\Omega \frac{\tilde{g}_\mu(\zeta)\bar{\partial}\tilde{g}_\nu(\zeta)}{\zeta - z}\, d\zeta\, d\bar{\zeta},$$

but it is not clear that the integral is convergent à priori, so instead we begin by viewing this formally as

$$\frac{1}{2\pi i} \iint_\Omega \frac{\tilde{g}_\mu(\zeta)\bar{\partial}\tilde{g}_\nu(\zeta)}{\zeta - z}\, d\zeta\, d\bar{\zeta} = \sum \frac{1}{2\pi i} \iint_{L^n_{J,j}} \frac{\tilde{g}_\mu(\zeta)\bar{\partial}\tilde{g}_\nu(\zeta)}{\zeta - z}\, d\zeta\, d\bar{\zeta},$$

and attempt to introduce additional cancellations in each term. We begin by estimating individual terms. For ease of notation, we define

$$I^n_{J,j}(z) = \frac{1}{2\pi i} \iint_{L^n_{J,j}} \frac{\tilde{g}_\mu(\zeta)\bar{\partial}\tilde{g}_\nu(\zeta)}{\zeta - z}\, d\zeta\, d\bar{\zeta}.$$

Let us suppose that (m, I) is such that $L^n_{J,j} = \hat{T}^n_J \cap \hat{T}^m_I$. We first note that, for $z \in \overline{L^n_{J,j}} \setminus \{x_1, x_2\}$,

$$\begin{aligned}
|I^n_{J,j}(z)| &\lesssim \iint_{L^n_{J,j}} \frac{|g_\nu^{(n,J)}\bar{\partial}\phi_{(n,J)} + g_\nu^{(m,I)}\bar{\partial}\phi_{(m,I)}|}{|\zeta - z|}\, dx\, dy \\
&\lesssim \iint_{L^n_{J,j}} \frac{|g_\nu^{(n,J)} - g_\mu^{(m,I)}||\bar{\partial}\phi_{(n,J)}|}{|\zeta - z|}\, dx\, dy \\
&\lesssim \frac{1}{\sigma_n \delta_n n^3} \iint_{L^n_{J,j}} \frac{dx\, dy}{|\zeta - z|} \\
|I^n_{J,j}(z)| &\lesssim \frac{1}{n^3},
\end{aligned}$$

where we have exploited the fact that $\sum \chi_{\hat{T}^n_J} \leq 2$ in the second line and the proposition in the third. Proceeding in a similar manner, we can achieve estimates

$$\left| \iint_{L^n_{J,j}} \tilde{g}_\mu \bar{\partial} g_\nu \, d\zeta \, d\bar{\zeta} \right| \lesssim \frac{\sigma_n \delta_n}{n^3}.$$

To generate further cancellations, we employ the "derivative matching trick" from rational approximation theory (see [32] or Chapter XII of [15]). To this end, we introduce functions $k^n_{J,j}$ defined as follows. Let $\tilde{K}^n_{J,j}$ be a "square" $K^{m'}_{I'} = K \cap Q^{m'}_{I'}$ with $\tilde{K}^n_{J,j} \cap L^n_{J,j} \neq \emptyset$ and side length comparable to $\sigma_n \delta_n$. The analytic capacity $\gamma(\tilde{K}^n_{J,j})$ is comparable to $\sigma_n \delta_n$, so if $f^n_{J,j}$ is the Ahlfors function for $\tilde{K}^n_{J,j}$ then, choosing (uniformly bounded) constants $c^n_{J,j}$ appropriately, and setting

$$k^n_{J,j}(z) = \frac{c^n_{J,j}}{n^3} f^n_{J,j}(z)$$

then $\|k^n_{J,j}\|_\infty \lesssim \frac{1}{n^3}$ and

$$(k^n_{J,j})'(\infty) = \frac{1}{2\pi i} \int_{L^n_{J,j}} \tilde{g}_\mu \bar{\partial} \tilde{g}_\nu \, d\zeta \, d\bar{\zeta}.$$

For the estimates that follow, we will employ the following form of Schwarz's lemma.

Lemma 2.2 (Schwarz's Lemma). *If E is a compact set and $f \in H^\infty(\mathbb{C}^* \setminus E)$ has a double zero at infinity then*

$$|f(z)| \lesssim \frac{\|f\|_\infty \mathrm{diam}(E)^2}{d(z,E)^2}.$$

Let us define

$$h^n_{J,j}(z) := I^n_J(z) - k^n_{J,j}(z).$$

Applied in the current context, the lemma yields

$$|h^n_{J,j}(z)| \lesssim \frac{1}{n^3} \left(1 \wedge \frac{(\sigma_n \delta_n)^2}{d(z, \tilde{K}^n_{J,j} \cup L^n_{J,j})^2} \right). \tag{3}$$

Since $\bar{\partial} k^n_{J,j} = 0$ in Ω, formally we have

$$\bar{\partial} \sum_{(n,J,j)} h^n_{J,j} = \tilde{g}_\mu \bar{\partial} \tilde{g}_\nu,$$

so it suffices to check the boundedness of the sum.

Fix $z \in \Omega$. For each $n \in \mathbb{N}$ there are at most boundedly many terms for which the minimum is one in the inequality (3), and the summability of $\frac{1}{n^3}$ shows that these terms, summed over n, give a contribution which is controlled by the sum $\sum \frac{1}{n^3}$.

For remaining terms, we distinguish between the cases $\sigma_n \gtrsim d(z,K)$ and $\sigma_n \lesssim d(z,K)$. Let $n_0 \in \mathbb{N}$ be such that $d(z,K) \asymp \sigma_{n_0}$ (the maximum principle prevails over any z with $d(z,K) \gg 1$). For $m \geq n_0$,

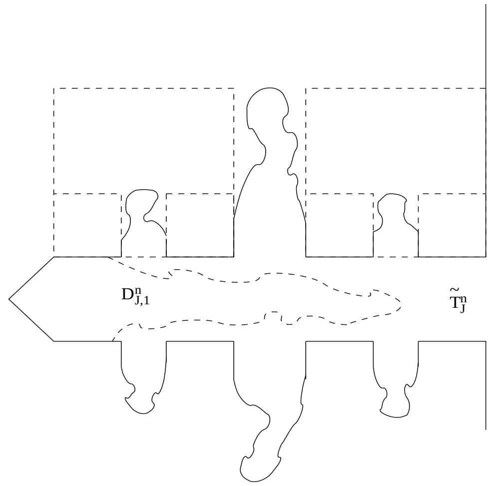

Figure 3: One of the regions $\tilde{T}^n_J$.

all remaining terms are at (roughly) distance σ_{n_0} or more away. Inductively one can show that there are at most 4^{m-n_0+k} terms having singularities a distance σ_{n_0-k} away from z, and so Schwarz lemma bounds the sum of these terms by a constant multiple of

$$\begin{aligned} \frac{1}{m^3}\sum_{k=0}^{n_0}\frac{\sigma_m^2\delta_m^2 4^{m-n_0+k}}{\sigma_{n_0-k}^2} &\leq \frac{1}{m^3}\sum_{k=0}^{n_0}\frac{4^m\sigma_m^2}{4^{n_0-k}\sigma_{n_0-k}^2} \leq \frac{1}{m^3}\sum_{k=0}^{n_0}1 \\ &\leq \frac{n_0}{m^3} \leq \frac{1}{m^2} \end{aligned}$$

Similarly, if $m < n_0$, there are 4^{k-m} terms at roughly distance σ_{m-k}, and so for these terms the sum is controlled by $m^{-3}\sum_0^m \frac{\sigma_m^2 4^{k-m}}{\sigma_{m-k}^2} \lesssim m^{-2}$. Summing these bounds over m, we obtain $\|\sum_{n,J,j} h^n_{J,j}\|_{L^\infty(\Omega)} \lesssim 1$ as desired. This proves the theorem. $\square$

3 Proof of Proposition 2.1

Let us now turn to the proof of Proposition 2.1. By normal families, it suffices to perform the construction in the domains $\{\hat{T}^n_J : |J| = n, n \leq N\}$, provided we obtain constants independent of N.

Rather than working directly with the sets $\hat{T}^n_J$, we will consider slightly enlarged domains $\tilde{T}^n_J$, defined as follows. We first define $\tilde{T}^N_J = \hat{T}^N_J$. Now let ϕ^N_J be a conformal mapping from $\tilde{T}^N_J$ to $\mathbb{D}$ preserving the symmetries of the domain $\tilde{T}^N_J$. Let $E_j = E_j(N, J)$ be the arc of $\partial\mathbb{D}$ defined by $E_j := \phi^N_J(\partial L^N_{J,j} \cap \partial\hat{T}^N_J)$, and let $E^*_j = 3E_j$. Defining $\theta_N = \pi(1 - \frac{c_1}{\log N})$, the constant c_1 to be determined later, let γ^*_j be the circular arc in $\mathbb{D}$ with endpoints coincident with those of E^*_j and intersecting at an angle of θ_N. Easy length-area/extremal length estimates in T^N_J yield $\omega(z^N_J, \ell^N_{J,j}, T^N_J) \lesssim e^{-\frac{c_0}{\delta_N}}$ for some constant c_0 not depending on N. Employing the maximum principle and following the mapping to the disc, we find

that the length of the arcs γ_j^* is $o(\frac{1}{\log N})$ by our assumptions on the sequence $\{\delta_n\}$. Now by elementary geometry, the disc determined by γ_j^* has radius r bounded by

$$r \lesssim \frac{c_1}{\log N} e^{-\frac{c_0}{\delta_N}},$$

so that $r = o(1)$. The arcs γ_j^*, $1 \le j \le 4$, are therefore (uniformly) hyperbolically separated, at least for N greater than some n_0. Let us now define $D_{J,j}^N$ to be the simply connected domain which is the pre-image under ϕ_J^N of the domain bounded by $E_j^* \cup \gamma_j^*$. We then define domains

$$\tilde{T}_I^{N-1} := \hat{T}_I^{N-1} \cup \bigcup \{D_{J,j}^N : L_{J,j}^N \subset \hat{T}_I^{N-1}\}$$

for each multi-index I of length $N-1$.

Proceeding inductively, let us suppose that the domains $\tilde{T}_J^n$ have already been constructed for $n > m \ge n_0$, and, fixing $(m+1, J)$, let ϕ_J^{m+1} be a conformal mapping from $\tilde{T}_J^{m+1}$ preserving the symmetries of the domain. Let $E_j = E_j(m+1, J)(\partial L_{J,j}^{m+1} \cap \partial \hat{T}_J^{m+1})$, let $E_j^* = 3E_j$, and let γ_j^* be the circular arc in $\mathbb{D}$ with endpoints coincident with those of E_j^* and intersecting at an angle of $\theta_{m+1} = \pi(1 - \frac{c_1}{\log(m+1)})$. As above, length-area estimates in T_J^{m+1} yield $\omega(z_J^{m+1}, \ell_{J,j}^{m+1}, T_J^{m+1}) \lesssim e^{-\frac{c_0}{\delta_{m+1}}}$, and thus the length of the arcs γ_j^* is $o(\frac{1}{\log(m+1)})$ by our assumptions on the sequence $\{\delta_n\}$. Moreover, the disc determined by γ_j has radius $r = r_m$ bounded by

$$r \lesssim \frac{c_1}{\log(m+1)} e^{-\frac{c_0}{\delta_{m+1}}}, \tag{4}$$

so that $r_m = o(1)$. The arcs γ_j^*, $1 \le j \le 4$, are therefore hyperbolically separated since we have taken $m \ge n_0$. Let us now define $D_{J,j}^{m+1}$ to be the pre-image under ϕ_J^{m+1} of the domain bounded by $E_j^* \cup \gamma_j^*$. We then define domains

$$\tilde{T}_I^m := \hat{T}_I^m \cup \bigcup \{D_{J,j}^n : n > m, L_{J,j}^n \subset \hat{T}_I^m\}$$

for each multi-index I of length m. (See Figure 3.) We note that for $n \ge n_0$ the estimate (4) yields that these domains are simply connected and also that $\tilde{T}_J^n \cap \tilde{T}_I^n = \emptyset$ when $I \ne J$.

For $n < n_0$ we will take

$$\tilde{T}_J^n = \hat{T}_J^n \cup \bigcup \{D_{I,j}^m : n \ge n_0, L_{I,j}^m \subset \hat{T}_J^n\}.$$

Proceeding from these definitions, we will construct our solutions from the top down, exploiting the natural generations structure of the Cantor set. At each stage we will obtain corona solutions $\{g_\mu^{(n,J)}\} \in H^\infty(\tilde{T}_J^n)$, and the corona solutions specified by the proposition will simply be the restrictions of these solutions to $\hat{T}_J^n$. For $n \le n_0$, we construct our corona solutions by applying the corona theorem for finitely connected domains to the domain $\Omega_{n_0} = \mathbb{C}^* \setminus [0,1]^2 \cup \bigcup_{n \le n_0} \bigcup_{|J|=n} \tilde{T}_{J,j}^n$. Given $n \ge n_0$, let us suppose we have already obtained the desired corona solutions $\{g_\mu^{(m,I)}\}$ in the regions $\tilde{T}_I^m$ for which $m < n$. As the domains $\{\tilde{T}_J^n\}_{|J|=n}$ are disjoint, we may construct our corona solutions in each of those domains separately.

Fix $\tilde{T}_J^n$. We note that $\tilde{T}_J^n \cap \left(\bigcup_{m<n} \bigcup_{|I|=m} \tilde{T}_I^m\right) = \bigcup_1^4 D_{J,j}^n$, and so in each region $D_{J,j}^n$ there are corona solutions constructed from previous generations, which we shall denote $\{g_\mu^j\}_\mu$. (Since $\sum_{n,J} \chi_{\tilde{T}_J^n} \le 2$, these solutions are uniquely determined among those previously constructed.) We now push the situation

forward to the unit disc according to the map ϕ_J^n. Let D_j denote the push-forward of the domain $D_{J,j}^n$, F_μ the push-forward of a corona datum f_μ, and G_μ^j the push-forward of g_μ^j. We note that D_j is a lens-shaped domain by construction.

In order to construct our corona solutions, we will, as above, reduce the problem to an appropriate $\bar\partial$-problem. Obtaining the desired bounds in this manner will require the use of interpolating sequences, so before continuing with the main line of argument we make a detour to obtain the special interpolating functions that we require.

Lemma 3.1. *There is a function $G \in H^\infty(\mathbb{D})$ of norm one with the following properties.*

1. *There is a constant c_2, not depending on n, such that $|G(z)| \geq c_2$ for $z \in \gamma_j^*$.*
2. *If $\tilde\alpha < \frac{\pi}{2}$ and L_j is the lens domain in $\mathbb{D}$ bounded by E_j and the circular arc γ_j meeting the endpoints $\zeta_1(j), \zeta_2(j)$ of E_j in angle $\tilde\alpha$ then*

$$|G(z)| \leq \frac{1}{n^3} \frac{d(z, \{\zeta_1, \zeta_2\})}{\operatorname{diam}(L_j)}.$$

Proof: We will obtain G by multiplying a number of outer functions. For the first factor, we define

$$H_0(z) = \exp\left\{-\int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z} 6\log n\, \chi_{\bigcup E_j^*}(\theta)\frac{d\theta}{2\pi}\right\},$$

noting that

$$\begin{aligned} |H_0(z)| &= \exp\left\{-\int_{\partial\mathbb{D}} P_z(\theta) 6\log n\, \chi_{\bigcup E_j^*}(\theta)\frac{d\theta}{2\pi}\right\} \\ &= \exp\left\{-6\log n\, \omega\left(z, \bigcup E_j^*, \mathbb{D}\right)\right\}, \end{aligned}$$

where P_z is the Poisson kernel. Due to our choice of angle θ_n, we have

$$\omega\left(z, \bigcup E_j^*, \mathbb{D}\right) \leq 4\omega(z, E_k, \mathbb{D}) \leq \frac{4c_1}{\log n}, \qquad z \in \gamma_k^*,$$

so that $|H_0(z)| \geq e^{-24c_1}$ on each γ_j^*. Also, on L_j we can easily compute that $\omega(z, E_j^*, \mathbb{D}) \geq \frac{1}{2}$, whereby $|H_0(z)| \leq \frac{1}{n^3}$ on L_j.

For the remaining factors we first note that on the imaginary axis the functions

$$u_\pm(z) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^- |e^{\pm i\frac{\pi}{6}} - e^{i\theta}| P_{iy}(\theta)\frac{d\theta}{2\pi}$$

are bounded below by $-\log 2$, and so after precomposing these functions by appropriate Möbius transformations and exponentiating, we obtain functions $\tilde H_{j,1}$, $\tilde H_{j,2}$ of norm one such that $|\tilde H_{j,1}(z)|, |\tilde H_{j,2}(z)| \geq \frac{1}{2}$ on each γ_k, $1 \leq k \leq 4$, and such that

$$\begin{aligned} |\tilde H_{j,1}(z)| &\leq \frac{|z-\zeta_1(j)|}{\operatorname{diam}(L_j)}, \\ |\tilde H_{j,2}(z)| &\leq \frac{|z-\zeta_2(j)|}{\operatorname{diam}(L_j)}, \end{aligned}$$

for $z \in L_j$. Setting $H_j = \tilde{H}_{j,1}\tilde{H}_{j,2}$, and then $G = \prod_0^4 H_j$ thus provides the desired function. $\square$

Fixing $\beta > 0$, if $\{z_k\} \in \gamma_j$ are points satisfying

$$|z_k - z_\ell| \geq \beta(1 - |z_k|), \qquad k \neq \ell, \tag{5}$$

then $\sum_k (1 - |z_k|)\delta_{z_k}$, where δ_{z_k} in this case denotes the unit point mass at z_k, is a Carleson measure with norm depending only on β, and so $\{z_k\}$ is an interpolating sequence with constant of interpolation depending only on β by Carleson's interpolation theorem for $H^\infty(\mathbb{D})$ [7]. Due to the hyperbolic separation of the arcs γ_j in $\mathbb{D}$, this remains true for a sequence $S = \bigcup_1^4 S_j$ with each S_j a sequence in γ_j satisfying this condition. Then if $G(z)$ is the function provided by Lemma 3.1, given $\{w_j\} \in \ell^\infty$ we can find an interpolating function $f \in H^\infty(\mathbb{D})$ with $f(z_j) = \frac{w_j}{G(z_j)}$. The function $g = fG \in H^\infty(\mathbb{D})$ then satisfies

$$g(z_j) = w_j,$$

$$\|g\|_{L^\infty(\mathbb{D})} \leq A' c_0,$$

$$|g(z)| \leq \frac{A' d(z, \{\zeta_1, \zeta_2\})}{n^3 \operatorname{diam}(L_j)}, \qquad z \in L_j,$$

for L_j and $\zeta_1(j), \zeta_2(j)$ defined as in the lemma, where A' bounds the largest constant of interpolation for a maximal sequence on $\bigcup_1^4 \gamma_j$ satisfying (5). Fixing β for the remainder of the proof, let $A = c_0 A'$.

Lemma 3.2. *Let $S = \{z_n\}$ be a maximal sequence on $\bigcup_1^4 \gamma_j$ satisfying*

$$|z_k - z_\ell| \geq \frac{1 - |z_j|}{8A^2}, \qquad k \neq \ell.$$

Then there are functions $h_j \in H^\infty(\mathbb{D})$ such that

$$h_j(z_j) = 1,$$

$$\|h_j\|_{L^\infty(\mathbb{D})} \leq A^2,$$

$$\sum_j |h_j(z)| \leq \frac{\log 8A^2}{\log \beta^{-1}} A^2,$$

and such that

$$\sum_j |h_j(z)| \leq \frac{\log 8A^2}{\log \beta^{-1}} \frac{A^2}{n^6} \left(\frac{d(z, \{\zeta_1, \zeta_2\})}{\operatorname{diam}(L_j)} \right)^2, \qquad z \in L_j,$$

where L_j and $\zeta_1(j), \zeta_2(j)$ are defined as in the Lemma 3.1.

Proof: This is a mild refinement of an argument of Varopoulos [31]. We note that the sequence S can be split into $\frac{\log 8A^2}{\log \beta^{-1}}$ disjoint sequences S_m such that (5) holds. Restricting our attention to a subsequence S_m, it suffices to consider the case that S_m is finite, $S_m = \{z_1, \ldots, z_{n_0}\}$, as one may then employ normal families to the construction below. Set $\omega = e^{\frac{2\pi i}{n_0}}$. Employing the remarks above, by Lemma 3.1 we may

choose $f_j \in H^\infty(\mathbb{D})$ such that $f_j(z_k) = \omega^{jk}$, $\|f_j\|_{L^\infty(\mathbb{D})} \le A$, and $|f_j(z)| \le \frac{A}{n^3}\frac{d(z,\{\zeta_1,\zeta_2\})}{\mathrm{diam}(L_\ell)}$ when $z \in L_\ell$ for each $1 \le \ell \le 4$. If we now define

$$h_j(z) = \left(\frac{1}{n_0}\sum_{k=1}^{n_0}\omega^{-jk} f_k(z)\right)^2,$$

then

$$h_j(z_i) = \left(\frac{1}{n_0}\sum_{k=1}^{n_0}\omega^{(i-j)k}\right)^2 = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

$$\|h_j\|_{L^\infty(\mathbb{D})} \le A^2,$$

and

$$|h_j(z)| \le \frac{A^2}{n^6}\left(\frac{d(z,\{\zeta_1,\zeta_2\})}{\mathrm{diam}(L_\ell)}\right)^2$$

for $z \in L_\ell$. Also,

$$\begin{aligned}\sum_{j=1}^{n_0}|h_j(z)| &= \frac{1}{n_0^2}\sum_{k=1}^{n_0}\sum_{j,\ell}\omega^{-jk}\omega^{j\ell} f_j(z)\bar{f}_\ell(z) \\ &= \frac{1}{n_0^2}\sum_{j=1}^{n_0} n_0|f_j(z)|^2,\end{aligned}$$

so that

$$\sum_{j=1}^{n_0}|h_j(z)| \le A^2$$

throughout the unit disc, and

$$\sum_{j=1}^{n_0}|h_j(z)| \le \frac{A^2}{n^6}\left(\frac{d(z,\{\zeta_1,\zeta_2\})}{\mathrm{diam}(L_\ell)}\right)^2,$$

for $z \in L_\ell$ as desired. □

With these lemmas in hand we turn now to constructing our special corona solutions. By Carleson's original corona theorem [8], there are corona solutions $\{g_\mu\}$ in $\mathbb{D}$ with $\|g_\mu\| \le c(M,\eta)$. To generate corona solutions close to G^j_μ on D_j our first instinct is to take a partition of unity and paste these together. In doing this we first add to the domains D_j to generate overlap. Specifically, let us define

$$\tilde{D}_0 = \left(\mathbb{D} \setminus \bigcup_1^4 D_j\right) \cup \bigcup_{z_j \in S} B\left(z_j, \frac{1-|z_j|}{4A^2}\right)$$

and

$$\tilde{D}_j = D_j \cup \bigcup_{z_k \in S\cap\gamma_j} B\left(z_j, \frac{1-|z_j|}{4A^2}\right)$$

for $1 \leq j \leq 4$. For ease of notation in what follows, we will denote the region of overlap, $\bigcup_1^4(\tilde{D}_j \cap \tilde{D}_0)$, by U. Let $\psi_0, \ldots, \psi_4$ be a partition of unity subordinate to $\tilde{D}_0, \ldots, \tilde{D}_4$ satisfying $|\nabla\psi_j(z)| \lesssim (1-|z|)^{-1}$. Our initial pasting is then

$$\tilde{g}_\mu := \psi_0 g_\mu + \sum_{j=1}^{4} \psi_j G_\mu^j.$$

To obtain a bounded analytic solution from this we now take an approach much like that above. In particular, we wish to find functions $a_{\mu\nu} \in L^\infty(\mathbb{D})$ such that

$$\bar{\partial} a_{\mu\nu} = \tilde{g}_\mu \bar{\partial} \tilde{g}_\nu$$

and

$$|a_{\mu\nu}(z)| \lesssim n^{-6} \left(\frac{d(z, \{\zeta_1, \zeta_2\})}{\operatorname{diam}(L_j)} \right)^2,$$

for $z \in L_j$, where L_j is the lens domain bounded by E_j and the circular arc meeting the endpoints $\zeta_1(j), \zeta_2(j)$ of E_j in angle $\tilde{\alpha} < \frac{\pi}{2}$.

Lemma 3.3. *If $B \in L^\infty(U)$ and $b(z) = \frac{B(z)}{1-|z|}\chi_U$ then there is $F \in L^\infty(\mathbb{D})$ such that*

$$\bar{\partial} F = b$$

in the sense of distributions on $\mathbb{D}$,

$$\|F\|_\infty \lesssim \|B\|_\infty,$$

and

$$|F(z)| \lesssim n^{-6} \left(\frac{d(z, \{\zeta_1, \zeta_2\})}{\operatorname{diam}(L_j)} \right)^2, \qquad z \in L_j,$$

where L_j is the lens domain bounded by E_j and the circular arc meeting the endpoints $\zeta_1(j), \zeta_2(j)$ of E_j in angle $\tilde{\alpha} < \frac{\pi}{2}$.

Proof: We follow an argument due to Peter Jones [20]. Let $\{h_m\}$ be the functions provided by Lemma 3.2, and let us write U as the disjoint union of sets $U_m \subset B(z_m, \frac{1-|z_m|}{4A^2})$, and let us define

$$F(\zeta) = \sum_m \frac{1}{\pi} \iint_{U_m} \frac{h_m(\zeta)}{h_m(z)} \frac{b(z)}{\zeta - z}\, dx\, dy.$$

Formally, $\bar{\partial} F(\zeta) = b(\zeta)$, so it suffices to check the convergence of the sum. Noting that $|h_m(\zeta)| \geq \frac{1}{2}$ in U_m by Schwarz's lemma, termwise estimates give

$$\begin{aligned}
\left| \frac{1}{\pi} \iint_{U_m} \frac{h_m(\zeta)}{h_m(z)} \frac{b(z)}{\zeta - z}\, dx\, dy \right| &\leq \frac{2}{\pi} |h_m(\zeta)| \iint_{U_m} \frac{|b(z)|}{|\zeta - z|}\, dx\, dy \\
&\leq \frac{1}{\pi} \|B\|_\infty |h_m(\zeta)| \frac{(1-|z_m|)^{-1}}{1-(4A^2)^{-1}} \iint_{B\left(z_m, \frac{1-|z_m|}{4A^2}\right)} \frac{dx\, dy}{|\zeta - z|} \\
\left| \frac{1}{\pi} \iint_{U_m} \frac{h_m(\zeta)}{h_m(z)} \frac{b(z)}{\zeta - z}\, dx\, dy \right| &\leq \frac{4}{3A^2} |h_m(\zeta)| \|B\|_\infty.
\end{aligned}$$

Summing, we find that

$$\|F\|_\infty \leq \frac{16}{3}\frac{\log 4A^2}{\log \beta^{-1}}\|B\|_\infty$$

and

$$|F(\zeta)| \leq \frac{16}{3}\frac{\log 4A^2}{\log \beta^{-1}} n^{-6}\|B\|_\infty \left(\frac{d(\zeta, \{\zeta_1, \zeta_2\})}{\mathrm{diam}(L_j)}\right)^2,$$

for $\zeta \in L_j$ due to the special properties of the functions h_m. □

Given this, the functions

$$G_\mu(z) = \tilde{g}_\mu(z) + \sum_{\nu=1}^{M}(a_{\mu\nu}(z) - a_{\nu\mu}(z))f_\nu(z),$$

are corona solutions in $\mathbb{D}$ satisfying

$$|G_\mu(z) - G_\mu^j(z)| \leq \sum_{\nu=1}^{M}|f_\nu(z)||a_{\mu\nu}(z) - a_{\nu\mu}(z)| \lesssim Mn^{-6}\left(\frac{d(z, \{\zeta_1, \zeta_2\})}{\mathrm{diam}(L_j)}\right)^2$$

on L_j. For fixed $\tilde{\alpha}$, if the angle α defining the lozenges $L^n_{J,j}$ in (1) is sufficiently small then $\phi^n_J(L^n_{J,j}) \subset L_j$ (reindexing as appropriate). Mapping back to the domains $\tilde{T}^n_J$, since $(\phi^n_J)^{-1}$ behaves as $(z - \zeta_i(j))^{\frac{1+\alpha}{2}}$ about $\zeta_i(j)$, and so we obtain

$$\begin{aligned} |g_\mu^{(n,J)}(z) - g_\mu^j(z)| &\lesssim n^{-6}\left(\frac{d(z, \{x_1, x_2\})}{\mathrm{diam}(L^n_{J,j})}\right)^{1+\alpha} \\ &\lesssim n^{-6}\frac{d(z, K)}{\mathrm{diam}(L^n_{J,j})}, \end{aligned}$$

on $L^n_{J,j}$, where $\{x_1, x_2\} = K \cap \overline{L^n_{J,j}}$, and $\|g_\mu^{(n,J)}\|_\infty \leq C(M, \eta)$, with constants uniform in (n, J).

This completes the proof of the proposition. □

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